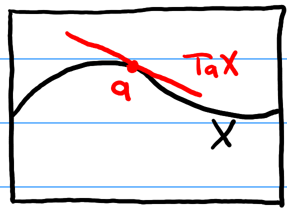
Smooth varieties

Big picture

- X variety, $a \in X \rightsquigarrow C_a X$ cone at a → dim = $\operatorname{codim}_X \{a\}$ → cut out by initial terms of defining equations

- in practice: nice to have linear space locally approximating X at a $\rightsquigarrow$ tangent space $T_a X$
  cut out by linear terms of defining equations



Def (Tangent spaces)

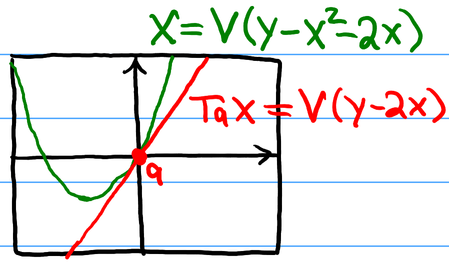
show: def. indep. of choices

X variety, $a \in X \rightsquigarrow$ choosing affine nbhd: X affine, $a = 0$

$$T_a X := V(f_1 : f \in I(X)) \subseteq \mathbb{A}^n \qquad \text{tangent space of } X \text{ at } a$$

$f_1 \in K[x_1,\dots,x_n]$ linear term; $f_0 = 0$ by assumption

Exa $X = V(y - x^2 - 2x)$
$\rightsquigarrow T_0 X = V(y - 2x)$



Rmks

(a) $I(X) = \langle S \rangle \rightsquigarrow T_a X = V(f_1 : f \in S)$ suff. to take lin. part of generators

Pf $f, g \in S$, $x \in T_a X$

$\rightsquigarrow (f+g)_1(x) = f_1(x) + g_1(x) = 0 + 0 = 0$

$(h \cdot f)_1(x) = h(0) \cdot \underbrace{f_1(x)}_{0} + \underbrace{f(0)}_{0} \cdot h_1(x) = 0. \quad \forall h \in K[x_1,\dots,x_n]$

$\Rightarrow V(f_1 : f \in S) \subseteq T_a X$ (other inclusion: easy). #

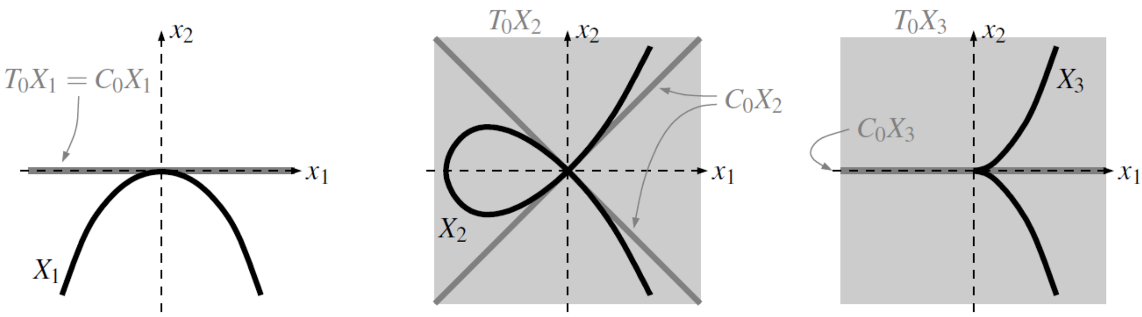
(b) Important to take $I(X)$:

$V(x) = V(x^2)$ but $\{0\} = V((x)_1) \neq V((x^2)_1) = V(0) = \mathbb{A}^1$

(c) $f \in I(X)$ with $f_1 \neq 0 \rightsquigarrow f_1 = f^{in}$ initial term

$\xRightarrow{[\text{Ex } 9.22(b)]}$ $\underbrace{C_a X}_{V(f^{in} \,:\, f \in I(X))} \subseteq \underbrace{T_a X}_{V(f_1 \,:\, f \in I(X))}$ and so $\dim T_a X \geq \operatorname{codim}_X \{a\}$.

Exa



$X_1 = V(x_2 + x_1^2)$ $\qquad$ $X_2 = V(x_2^2 - x_1^2 - x_1^3)$ $\qquad$ $X_3 = V(x_2^2 - x_1^3)$

C_0X_1 and T_0X_1 agree

$C_0X_2 \subseteq T_0X_2$ forces $T_0X_2 = \mathbb{A}^2$

$C_0X_3 =$ lin. space of dim 1 but still $T_0X_3 = \mathbb{A}^2$

Intrinsic description of Zariski tangent space

Note Defined $T_a X$ assuming $X \subseteq \mathbb{A}^n$, $a=0$

$\rightsquigarrow$ How do we know its dimens. is indep. of choices?

Goal more intrinsic descript. in terms of rings of functions on X.

Lem $X \subseteq \mathbb{A}^n$ affine variety, $a = 0 \in X$, $I(a) = I_X(a) = \langle \bar{x}_1, \dots, \bar{x}_n \rangle \trianglelefteq A(X)$

$$\Rightarrow \quad I(a)/I(a)^2 \cong \mathrm{Hom}_K(T_a X, K)$$ natural isom. of K-vector spaces

$\rightsquigarrow T_a X \cong (I(a)/I(a)^2)^\vee$ dual vector space

Pf. Consider K-linear map

$$\varphi : I(a) \longrightarrow \mathrm{Hom}_K(T_a X, K), \quad \bar{f} \longmapsto f_1|_{T_a X}$$

- Well-defined: $\bar{f} = \overline{f+h}$, $h \in I(X) \rightsquigarrow h_1|_{T_a X} = 0$ by def.
- surjective: $g \in \mathrm{Hom}_K(T_a X, K) \rightsquigarrow \bar{g} \in I(a)$, $g = g_1$

Isomorphism thm: suffices to show $\mathrm{Ker}(\varphi) = I(a)^2$.

"$\supseteq$": $\bar{f}, \bar{g} \in I(a) \Rightarrow (f \cdot g)_1 = \underbrace{f(0)}_{0} g_1 + \underbrace{g(0)}_{0} \cdot f_1 = 0 \Rightarrow \varphi(\bar{f}\bar{g}) = 0$.

"$\subseteq$": $\underset{\text{dim } k}{W} = \{ g_1 : g \in I(X) \} \subseteq \underset{\text{dim } n}{\mathrm{Span}_K} \{x_1, \dots, x_n\} = V = (K^n)^\vee$

$\rightsquigarrow T_a X = $ zero locus of W $\leftarrow$ dim $n-k$

$\rightsquigarrow (T_a X)^\perp = \{ h \in V : h|_{T_a X} = 0 \}$ $\leftarrow$ dim k, contains $W \Rightarrow (T_a X)^\perp = W$

$\bar{f} \in \mathrm{Ker}(\varphi) \rightsquigarrow f_1 \in (T_a X)^\perp \rightsquigarrow \exists g \in I(X)$ with $g_1 = f_1$

$\Rightarrow f - g$ has no const. or linear term

$\Rightarrow \bar{f} = \overline{f-g} \in I(a)^2$. $\square$

$\uparrow$ polyn. h with $h_0 = h_1 = 0$

Problem description still relies on affine nbhd. (for $A(X)$)
Solution reformulate using local ring $\mathcal{O}_{X,a}$

Recall: $\mathcal{O}_{X,a} = S^{-1}A[X]$ with $S = A(X) \setminus I(a)$
local ring with maximal ideal

$$I_a = S^{-1}I(a) = \left\{ \frac{g}{f} : g, f \in A(X) \text{ with } g(a)=0,\ f(a)\neq 0 \right\}$$

Cor With notations as above:

$$I(a)/I(a)^2 \cong (S^{-1}I(a))/(S^{-1}I(a))^2$$

$\rightsquigarrow$ X variety, $a \in X$, $I_a \trianglelefteq \mathcal{O}_{X,a}$ max. ideal

$\Rightarrow$ $T_aX \cong (I_a/I_a^2)^\vee$ (Zariski) tangent space

independent of choices

Pf Want to show: multiplying by $f \in S$ is invertible on $I(a)/I(a)^2$ (*)

Then:

$$I(a)/I(a)^2 \overset{(*)}{\cong} S^{-1}(I(a)/I(a)^2) \cong S^{-1}I(a)/(S^{-1}I(a))^2$$

localization commutes with quotients [G, Cor 6.22(b)]

$f \in S \rightsquigarrow f(a) \neq 0 \in A(X)/I(a) \cong K$
$\rightsquigarrow 1/f = c \in A(X)/I(a)$ $\leftarrow$ take $c = 1/f(a)$

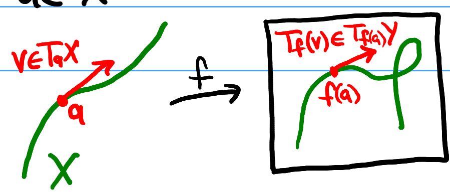
$\Rightarrow$ For $g \in I(a)/I(a)^2$ we have
$g/f = c \cdot g \in I(a)/I(a)^2$

not just $A(X)$-module, but $A(X)/I(a)$-module

$\Rightarrow$ Proves (*)

$\square$

Exercise $f: X \to Y$ morphism of varieties, $a \in X$
$\Rightarrow$ get linear map $T_f : T_aX \to T_{f(a)}Y$.



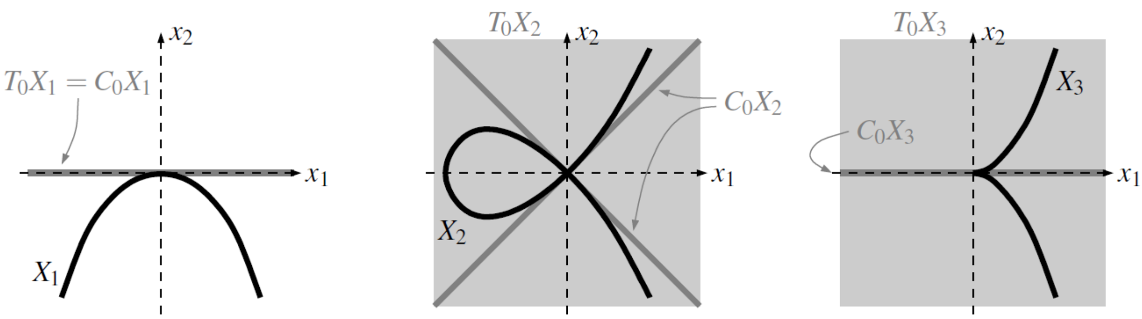
Smoothness and singular points

Have seen Both $C_a X \subseteq T_a X$ approximate X at $a \in X$

Good case $C_a X = T_a X$ ⇝ X can be approx. by lin. space at a

Def (Smooth and singular varieties)
X variety
(a) $a \in X$ is called a smooth, regular or non-singular point if $C_a X = T_a X$. Otherwise a is a singular point of X.
(b) X is singular if it has a singular point. Otherwise X is smooth, regular or non-singular.

Exa



$X_1 = V(x_2 + x_1^2)$ $\qquad$ $X_2 = V(x_2^2 - x_1^2 - x_1^3)$ $\qquad$ $X_3 = V(x_2^2 - x_1^3)$

Smooth (at 0) $\qquad$ Singular (at 0)

Lem X variety, $a \in X$, then the following are equivalent:
(a) a smooth in X
(b) $\dim T_a X = \operatorname{codim}_X \{a\}$
(c) $\dim T_a X \leq \operatorname{codim}_X \{a\}$

Proof $\dim C_a X = \operatorname{codim}_X \{a\}$ $\implies$ (a ⇒ b) and (b ⇒ c) clear

(c ⇒ a) $\underset{\text{dim} = \operatorname{codim}_X\{a\}}{C_a X} \subseteq \underset{\text{irreducible, dim} \leq \operatorname{codim}_X\{a\}}{T_a X}$ $\implies$ $C_a X = T_a X$. $\square$

Rmk (Smoothness in commutative algebra) X variety, $a \in X$

(a) $\mathcal{O}_{X,a}$ local ring with maximal ideal I_a

$\rightsquigarrow$ a smooth $\iff$ $\underbrace{\dim I_a/I_a^2 \overset{?}{=} \operatorname{codim}_X \{a\}}_{\text{purely a property of } \mathcal{O}_{X,a} \text{ : regular local ring}}$

$\uparrow$ $= \text{Krull-dimension}(\mathcal{O}_{X,a})$

(b) regular local ring $\Rightarrow$ integral domain

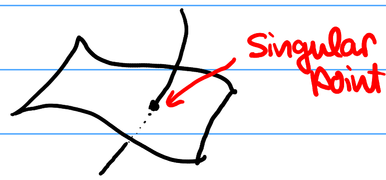
$\rightsquigarrow$ X is locally irreducible at $a \in X$:

$\uparrow$ a lies in unique irred. comp. X_i of X

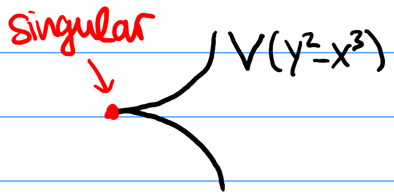
In this case: $\boxed{\operatorname{codim}_X \{a\} = \dim X_i}$

$\rightsquigarrow$ conversely: X_i, X_j different irred. components

$\Rightarrow$ any pt. in $X_i \cap X_j$ is singular



Note locally irred. not suffic. for smoothness:



Jacobi criteria

Q How to determine the smooth / singular pts. in practice?

Pro (Affine Jacobi criterion)
$X \subseteq \mathbb{A}^n$ affine variety, $a \in X$, $I(X) = \langle f_1, \dots, f_r \rangle$
Then X is smooth at a if and only if the Jacobi matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{\substack{i=1,\dots,r \\ j=1,\dots,n}} \in \mathrm{Mat}(r \times n, K)$$

has $\qquad \mathrm{rank} \geq n - \mathrm{codim}_X \{a\}$.
In this case: $\mathrm{rk}(J) = n - \mathrm{codim}_X \{a\}$.

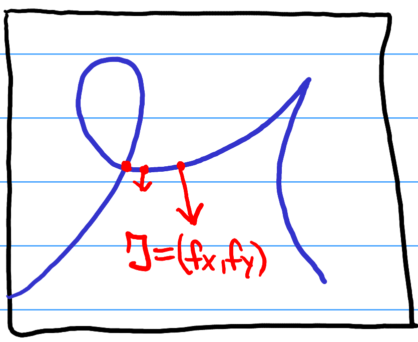
Pf $y_j = x_j - a_j$ shifted coordinates

formal Taylor expansion

$\rightsquigarrow$ linear term of f_i at a $= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(a) y_j$

$\Rightarrow T_a X = \mathrm{Ker}(J)$
Lem: a smooth $\Leftrightarrow \dim T_a \overset{(*)}{\leq} \mathrm{codim}_X \{a\}$
$\Leftrightarrow \mathrm{rk}(J) \geq n - \mathrm{codim}_X \{a\}$
Lem: inequality $(*)$ implies equality "=". $\square$

Exa



$X \subseteq \mathbb{A}^2$ with $I(X) = \langle f \rangle$
curve

X smooth at $a \in X$ if

$$\mathrm{rk} \underbrace{\left(\frac{\partial f}{\partial x}(a) \; \frac{\partial f}{\partial y}(a) \right)}_{J} \geq 1$$

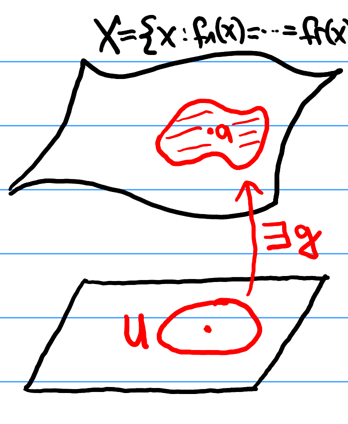
$\Leftrightarrow J \neq 0$

Exercise (Projective Jacobi criterion, Ex. 10.13)
$X \subseteq \mathbb{P}^n$ projective variety, $I(X) = \langle f_1, \dots, f_r \rangle$, f_i homog.
Then:
$$a \in X \text{ smooth} \iff \operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{\substack{i=1,\dots,r \\ j=0,\dots,n}} \geq n - \operatorname{codim}_X\{a\}$$

Downside of Jacobi criteria above: need generators $f_1, \dots, f_r$ of $I(X)$
Cor (Variants of the Jacobi criterion)
$f_1, \dots, f_r \in K[x_1, \dots, x_n]$ with $X = V(f_1, \dots, f_r) \subseteq \mathbb{A}^n$ and $a \in X$
(a) If $\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j} \geq n - \operatorname{codim}_X\{a\}$ then X is smooth at a
(b) If $\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j} = r$, then X is smooth at a, $\operatorname{codim}_X\{a\} = n - r$
Jacobi matrix has max. row rank.

Pf (a) Extend $f_1, \dots, f_r \in I(X)$ to gen. set $f_1, \dots, f_s$ of $I(X)$ $(s \geq r)$
$$\Rightarrow \operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{i=1,\dots,s \\ j=1,\dots,n}} \geq \operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{i=1,\dots,r \\ j=1,\dots,n}} \geq n - \operatorname{codim}_X\{a\}$$
$\rightsquigarrow$ Claim follows from affine Jacobi criterion
(b) Krull's princ. ideal thm $\Rightarrow$ all comp. X_i of X have $\dim \geq n-r$
$\Rightarrow \operatorname{codim}_X\{a\} = \max\{\dim X_i : a \in X_i\} \geq n - r$
$\rightsquigarrow \operatorname{rk}(J) \underset{\text{assumpt.}}{\geq} r \geq n - \operatorname{codim}_X\{a\} \Rightarrow X$ smooth at (a) & equality holds □

Rmk (Relation to implicit function theorem)



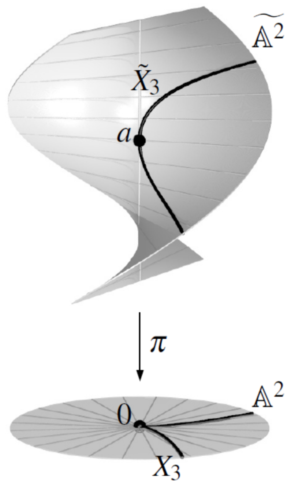
$f_1, \dots, f_r : \mathbb{R}^n \to \mathbb{R}$ $\mathcal{C}^1$-functions with
$$\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j} = r$$
$\Rightarrow$ solution set $X = \{x \in \mathbb{R}^n : f_i(x) = 0 \ \forall i\}$ is locally around a the graph of a $\mathcal{C}^1$-function $g : \mathbb{R}^{n-r} \supseteq U \to \mathbb{R}^r$.
also submanifold of dim $n-r$

Note f_i polynom. $\not\Rightarrow g$ polynomial
$f(x,y) = x^2 - y^3 \rightsquigarrow x = y^{3/2}$, $y = x^{2/3}$ not algebraic
smooth at $(1,1)$

Applications of the Jacobi criterion

Exa (Resolution of singularities)

$X_3 = V(\underbrace{x_2^2 - x_1^3}_{f}) \subseteq \mathbb{A}^2_{\mathbb{C}}$



$\Rightarrow J = \left(\frac{\partial f}{\partial x_1} \ \ \frac{\partial f}{\partial x_2}\right) = \left(-3x_1^2 \ \ 2x_2\right)$

$\mathrm{rk}(J) = \begin{cases} 0 & , (x_1,x_2) = (0,0) \\ 1 & , \text{otherwise} \end{cases}$

$n - \mathrm{codim}_{X_3}\{p\} = 2 - 1 = 1 \quad \forall p \in X$

Jacobi criterion: X_3 smooth at all pts p except at $p = (0,0)$.

$\tilde{X}_3 \longrightarrow X_3$ blow-up at 0 $\rightsquigarrow$ exc. locus $= \{a\} \in \tilde{X}_3$

Coordinates $((x_1,x_2),(y_1:y_2))$ on $\widetilde{\mathbb{A}}^2 \subseteq \mathbb{A}^2 \times \mathbb{P}^1$

↖ $x_1y_2 = x_2y_1$

Chart $U_1 = \{y_1 \neq 0\} \cong \mathbb{A}^2$ has coord. (x_1, y_2) ← $y_1 = 1$, $x_2 = x_1y_2$

$\rightsquigarrow \pi^{-1}(X_3) = V(x_2^2 - x_1^3) = V((x_1y_2)^2 - x_1^3) = V(x_1^2) \cup \underbrace{V(y_2^2 - x_1)}_{= \tilde{X}_3 \cap U_1}$

$\Rightarrow \tilde{X}_3$ smooth at $a = ((0,0),(1:0))$

↖ $\tilde{X}_3 \to X_3$ is <u>resolution of singularities</u>

More generally: blowing up singularities tends to make them "nicer".

Conjecture (Resolution of singularities)

X complete irreduc. variety. Then $\exists$ smooth, complete variety X' together with a birat'l map $X' \xrightarrow{f} X$.

↖ f defined on all of X'

Proven by Hironaka (1964) if char $K = 0$ and for arbitrary K if $\dim X \leq 3$ $\rightsquigarrow$ open in general!

Cor The set $X^{sm} \subseteq X$ of smooth pts. of a variety X is open.

Pf $a \in X$ smooth $\rightsquigarrow$ Want to show: a has open nbhd. of smooth pts.
$\rightsquigarrow$ restrict to affine open containing a: wlog: $X \subseteq \mathbb{A}^n$
Have seen: a smooth $\rightsquigarrow \mathcal{O}_{X,a}$ regular loc. ring $\rightsquigarrow X$ loc. irred. at a
$\rightsquigarrow$ restrict to irred. nbhd. of a
Can assume $X \subseteq \mathbb{A}^n$ irred. $\Rightarrow \operatorname{codim}_X\{p\} = \dim X \quad \forall p \in X$
Jacobi criterion: $X^{sm} = \{p \in X : \operatorname{rk}(J_p) \geq n - \dim X\}$

Jacobian matrix at p — open condition (one $(n-\dim X)$-minor of J_p does not vanish) $\square$

Rmk (Generic smoothness)
Fact $X^{sm} \subseteq X$ always dense. not easy to show! [Hartshorne, Thm I.5.3]

Exa $X = V(f) \subseteq \mathbb{A}^n$ for $f \in K[x_1,\dots,x_n]$ non-const., irreducible
Show: $X^{sm} \neq \emptyset$ $\rightsquigarrow$ then: dense since X irreducible
Otherwise: $\frac{\partial f}{\partial x_i}(a) = 0 \quad \forall a \in X \rightsquigarrow \frac{\partial f}{\partial x_i} \in I(X) = \langle f \rangle$ (Nullstellensatz)
f irred., $\deg_{x_i} \frac{\partial f}{\partial x_i} < \deg_{x_i} f$
Only possible if $\frac{\partial f}{\partial x_i} = 0 \quad \forall i = 1,\dots,n$

$\operatorname{char}(K) = 0 \Rightarrow f$ has deg. < 1 in all $x_i \Rightarrow f$ constant $\lightning$
$\operatorname{char}(K) = p \Rightarrow f$ polynomial in $x_1^p,\dots,x_n^p$
$f = g(x_1^p,\dots,x_n^p) = g(x_1,\dots,x_n)^p$ $\lightning$ to f irreducible
"Freshman's dream" e.g. $(x_1+x_2)^p = x_1^p + x_2^p$ in $\mathbb{F}_p[x_1,x_2]$ $\square$

Exa (Fermat hypersurface) $n,d \in \mathbb{N}_{>0}$
$\rightsquigarrow X = V_P(x_0^d + \dots + x_n^d) \subseteq \mathbb{P}^n$ Fermat hypersurface
Claim X is smooth $(\forall d,n,K)$.
Pf $J = (d x_0^{d-1} \; d x_1^{d-1} \dots d x_n^{d-1})$. • $\operatorname{char} K \nmid d \Rightarrow \operatorname{rk}(J) = 1$ at all pts of $\mathbb{P}^n$
• $p = \operatorname{char}(K) \mid d \rightsquigarrow d = d' \cdot p^r$, $p \nmid d'$
$\rightsquigarrow X = V_P((x_0^{d'} + \dots + x_n^{d'})^{p^r}) = V_P(x_0^{d'} + \dots + x_n^{d'})$ smooth by case above. $\square$